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CONVERGENT SHOCK WAVE IN AN IDEAL ELASTIC NONHOMOGENEOUS MEDIUM

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The boundary-value problem for symmetric focusing of a shock wave in a medium with variable density under a constant load (model of a porous body with variable initial velocity) is solved. The solution asymptotic is studied. Focusing in a homogeneous medium has been previously studied [1]. One inverse problem related to the choice of the optimal pressure conditions is examined. Constraints on the applicability of the model are touched on.

Suppose a uniform load $p_0(t)$ is applied to the surface of a sphere (cylinder, layer) whose initial density is a differentiable function of the radius $[\rho = \rho(r)]$ at a moment of time $t = 0$. We assume that the load instantaneously attains a finite value $p_0(t) > 0$ and does not increase any further (the physical meaning of this condition is that of an explosion on the surface); the medium is ideal (without tangential stresses). The density of the medium at any point ρ_1 is set equal to a constant $(0 < \rho < \rho_1)$ and remains constant if the pressure at this point reaches values arbitrarily greater than zero. This highly simplified model approximately describes the behavior of a body with variable porosity and uniform skeleton at high loads.

A shock wave will propagate from the surface to the center. The focusing process for the shock wave in a homogeneous medium has been studied in [1]. The purpose of the current report is to investigate the influence of nonhomogeneity on the motion of the medium behind the front of a convergent shock wave. In particular, the variation in the degree of cumulation of a shock wave is of some interest. It may be expected that, as in the case of an ideal gas of variable density [2], the choise of $\rho(r)$ can either weaken or intensify accumulation.

The following motion and continuity equations hold within the region bounded by the moving surface $r = R_1(t)$ and the shock wave front $r = R(t)$:

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$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+\frac{1}{\rho_1}\frac{\partial p}{\partial r}=0,\quad \frac{\partial ur^{\nu}}{\partial r}=(R_1(t)<r<R(t)), \tag{1}$$

and the following conditions hold at the shock-wave front and on the surface:

$$u=u_*(t)=\theta(R)\dot{R},\ p=p_*(t)=\rho(R)\theta(R)\dot{R}^2\ \ (r=R(t)), \tag{2}$$

$$P=P_0(t)\ \ (r=R_1(t)),$$

where U is mass velocity; $\nu=0, 1, 2$, which corresponds to the cases* of a layer, cylinder, and sphere; $\theta(R)=1-\rho(R)/\rho_1$; the dot above a variable denotes the time derivative; and $R(0)=R_1(0)=R_0$.

The continuity equations and the first condition on the front imply that

$$u=\theta(R)\dot{R}\left(\frac{R}{r}\right)^{\nu};\ R_1^{\nu+1}=R_0^{\nu+1}+(\nu+1)\int\limits_{R_0}^{R}\theta(y)\,y^{\nu}dy. \tag{3}$$

We substitute the equations for u in the first equation of (1) and carry out integration from $r=R$ to $r=R_1$, arriving at the following equation for R(t):

$$R\ddot{R}+\frac{1}{2}A_{\nu}\dot{R}^2=B_{\nu}'\ \ (R(0)=R_0,\dot{R}^2(0)=p_0(0)/[\rho(R_0)\theta(R_0)]); \tag{4}$$

$$A_0=2\left[R(\ln\theta)_R'-\frac{1-\theta}{R_1/R-1}\right];\quad B_0'=-\frac{p_0(t)}{\rho_1\theta(R)(R_1/R-1)};$$

$$A_1=2\left[1+R(\ln\theta)_R'-\frac{1-(\theta/2)\left(1+R^2/R_1^2\right)}{\ln(R_1/R)}\right];\ B_1'=-\frac{p_0(t)}{\rho_1\theta(R)\ln(R_1/R)};$$

$$A_2=2\left[2+R(\ln\theta)_R'-\frac{1-(\theta/2)\left(1+R^4/R_1^4\right)}{1-R/R_1}\right];\ B_2'=-\frac{p_0(t)}{\rho_1\theta(R)(1-R/R_1)}.$$

We introduce the dimensionless variables $x=R/R_0$ and $g=\dot{R}^2/\dot{R}^2(0)$, denoting by the vinculum variables as given by their values when $t=0$ or $x=1$.

Let us assume that p_0 is given as a function of the front radius x [1]. By solving the problem, we may then determine the function $p_0(t)$ which corresponds to the resulting solution (semiinverse method). A solution for any continuous function $p_0(t)$ can in all likelihood be constructed by successive approximations. Such a method of defining the boundary condition will not play a role in studying the asymptotic behavior of the solution as $x\to 0$ [1].

We let all the functions depend on t in place of x, without varying the notation. Equation (4) takes the form

$$x\frac{dg}{dx}+A_{\nu}g=B_{\nu}\quad (g(1)=1). \tag{5}$$

Here $B_{\nu}=-Q(x)/\varphi_{\nu}(x_{\nu}, x)$, where

$$\varphi_0=x_0/x-1,\ \varphi_1=\ln(x_1/x),\ \varphi_2=1-x/x_2;$$

$$Q(x)=2\theta(1)[1-\theta(1)]\overline{p_0}(x)/\theta(x);$$

$$x_{\nu}^{\nu+1}=1+(\nu+1)\int\limits_1^x\theta(y)\,y^{\nu}dy.$$

Analogously [1], we isolate the singularities in the improper integrals in the solution (5), obtaining

$$g=\frac{\exp[-G_{\nu}(x)]}{\theta^2(x)\psi_{\nu}^2(x)}\int\limits_1^x\theta^2(\xi)\psi_{\nu}^2(\xi)B_{\nu}(\xi)\exp[G_{\nu}(\xi)]\frac{d\xi}{\xi}\quad (0<x<1). \tag{6}$$

Here

$$G_0=2\int\limits_1^x\left[\frac{1}{1-\xi}-\xi_0\frac{1-\theta(\xi)}{1-\xi_0}\right]\frac{d\xi}{\xi},$$

$$G_1=\int\limits_1^x\left[\frac{2-\theta(\xi)\left(1+\xi_1^2\right)}{\ln\xi_1}-\frac{2}{\ln\xi}\right]\frac{d\xi}{\xi},$$

$$G_2 = \int_1^x \left[\frac{2}{1-\xi} - \frac{2-\theta(\xi)\left(1+\xi_2^4\right)}{1-\xi_2}\right]\frac{d\xi}{\xi},$$

where

$$\xi_\nu = \xi/\left[1+(\nu+1)\int_1^\xi \theta(y)\, y^\nu dy\right]^{\frac{1}{\nu+1}};$$

$$\psi_0 = x^{-1}-1,\quad \psi_1 = x\ln x,\quad \psi_2 = x(1-x).$$

Equations for the variables at the front are implied by Eq. (2),

$$\bar{p}_* = \bar{\rho}(x)\bar{\theta}(x) g(x);\quad \bar{u}_* = \bar{\theta}(x)\sqrt{g(x)}. \tag{7}$$

The mass velocity behind the front has the form, in accordance with Eq. (3),

$$u = u_*(x/z)^\nu \quad (z = r/R_1),$$

while $\bar{p}(x, z)$ will be determined by integrating the first equation of Eqs. (1) from some point z within the region to z = x,

$$\bar{p} = \bar{p}_* - \left[\left(xg\bar{\theta}' + \frac{1}{2}g'\bar{\theta} + \nu g\bar{\theta}\right)\varphi_\nu(z, x) - \frac{1}{2}g\left(1 - x^{2\nu}/z^{2\nu}\right)\theta\bar{\theta}\right]\left[1-\theta(1)\right].$$

Here the prime denotes differentiation with respect to x.

The function x = x(t) can be determined from the equation

$$t = \frac{R_0}{\dot{R}(1)}\int_x^1 \frac{dy}{\sqrt{g(y)}}.$$

Then the desired functions will be functions of the variables z and t.

We asymptotically calculate g(x) as $x \to 0$ using Eq. (6),

$$g \sim \frac{1}{\theta^2}\ (\nu = 0),\quad g \sim \frac{s^{\theta_0 - 2}(x)}{x^2\theta^2}\ (\nu = 1),\quad g \sim \frac{1}{x^{2+\theta_0}\theta^2}\ (\nu = 2), \tag{8}$$

$$\left(\theta_0 = \theta(0),\quad s = \ln\frac{x_1(0)}{x},\quad x_1(0) = \left[1 - 2\int_0^1 \theta(y)\, y\, dy\right]^{1/2}\right).$$

Equations (7) imply that the variables are asymptotic at the front. If θ and ρ do not simultaneously vanish when x = 0, the asymptotic behavior of all the functions as $x \to 0$ will not differ from the case of a homogeneous medium. Suppose that the density distribution $\rho/\rho_1 \sim x^\alpha$ ($\alpha > 0$) as $x \to 0$. Then $\theta \sim 1$ and

$$p_* \sim x^\alpha,\ \dot{R} \sim 1,\ u_* \sim 1\ (\nu = 0), \tag{9}$$

$$p_* \sim \frac{s^{-1}}{x^{2-\alpha}},\quad \dot{R} \sim \frac{s^{-1/2}}{x},\quad u_* \sim \frac{s^{-1/2}}{x}\ (\nu = 1),$$

$$p_* \sim \frac{1}{x^{3-\alpha}},\quad \dot{R} \sim \frac{1}{x^{3/2}},\quad u_* \sim \frac{1}{x^{3/2}}\ (\nu = 2).$$

The increment asymptotic equation for the specific internal energy at the front [$e = \frac{1}{2}R^2(1)\theta^2 g$], which here is an increment of the thermal internal energy and determines the temperature distribution in the medium [1], has the form

$$e_* \sim 1\ (\nu = 0),\ e_* \sim x^{-2}s^{-1}\ (\nu = 1),\ e_* \sim x^{-3}\ (\nu = 2).$$

Let us now consider the case $\theta \sim x^\beta$ as $\rho/\rho \to 1$ ($x \to 0$, $\beta > 0$)

$$p_* \sim x^{-\beta},\ \dot{R} \sim x^{-\beta},\ u_* \sim 1,\ e_* \sim 1\ (\nu = 0), \tag{10}$$

$$p_* \sim \frac{s^{-2}}{x^{2+\beta}},\quad \dot{R} \sim \frac{s^{-1}}{x^{1+\beta}},\quad u_* \sim \frac{s^{-1}}{x},\quad e_* \sim \frac{s^{-2}}{x^2}\ (\nu = 1),$$

$$p_* \sim \frac{1}{x^{2+\beta}},\quad \dot{R} \sim \frac{1}{x^{1+\beta}},\quad u_* \sim \frac{1}{x},\quad e_* \sim \frac{1}{x^2}\ (\nu = 2).$$

Equations (9) and (10) imply that the asymptotic formula for the equations u_* and e_* weakly depend on $\rho(x)$. They have a singularity as $x \to 0$ for $\nu = 1, 2$, whose order of magnitude varies slightly as we pass

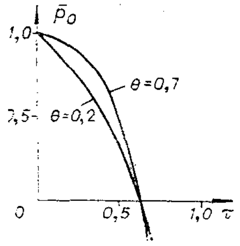

Fig. 1

from the case of zero density at the center to the case of zero porosity at the center. Nonhomogeneity exerts a strong effect on the behavior of p_*. When $\rho/\rho_1=x^3$ ($\nu=2$) and $\rho/\rho_1 \sim x^2 s(x)$ ($\nu=1$), pressure at the front $p_* \sim$ const. If the degree of decrease of density is greater or less than that indicated, pressure will either decrease or increase as the wave approaches the center. It is of interest to note that the degree of increase of p_* ($\nu=2$) is identical in Eqs. (9) and (10) when $\alpha+\beta=1$.

Our results may also be of interest for such applications as dynamic extrusion of metallic powders, in particular, for determining optimal extrusion conditions that can be formulated in the form of an inverse problem, for example, the problem of determining the form and magnitude of an applied momentum to obtain given extrusion conditions. Let us consider one particular case of this problem for a homogeneous cylindrical sample (ρ =const). We determine $p_0(t)$ under the condition that $p_*(t)\equiv\text{const}=p_0(t)$ (uniform extrusion of the sample). Then $\dot{R}(t)$ =const and $g\equiv 1$. Equation (5) turns into an equation for determining $p_0(x)$ and x plays the role of dimensionless time $(x+1+\dot{R}t/R_0)$.

We obtain $\bar{p}_0=[1+\ln\tilde{x}-(\theta/2)(1+\tilde{x}^2)]/(1-\theta)$. Here $\tilde{x}=x\ (1-\theta+\theta x^2)^{1/2}$.

Curves describing the dependence $\bar{p}_0=\bar{p}_0(\tau=-\dot{R}t/R_0)$ for $\theta=0.2$ and 0.7 are shown in Fig. 1. Clearly, negative pressures must be applied when $\tau>0.63$ ($x<0.37$) in order to maintain p_* constant. We note that the point $x\approx 0.37$ is the point at which p_* begins to grow in a homogeneous medium for any $p_0(x)$. If pressure p_0 remains zero when $\tau>0.63$, the cylinder will be compressed under identical conditions to about five-sevenths of its mass. The choice of $\rho(x)$ can theoretically ensure completely uniform compression.

In conclusion, let us make a number of remarks concerning the limits of applicability of the model of a permeable body to actual media. It has been noted [1] that one condition for the applicability of the model is $\dot{R}^2/c^2 \ll 1$, where c is the speed of sound behind the front of the shock wave. This condition is clearly violated as $\theta\to 0$ (here $|\dot{R}|$ is of the order of magnitude of c) and in the case of focusing (since $|\dot{R}|$ increases with increasing wave amplitude more rapidly than the speed of sound). Thus, the degree of cumulation will vary due to the compressibility of a skeleton with increasing p_* in the direction of the degree of cumulation of solid matter and will not be as great as that predicted by theory. The asymptotic equations (8)-(10) and previous [1] results must therefore be considered as upper limits of actual cumulation processes of shock waves in a porous medium.

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